13.4 Nonlinear Systems

In this final section, we want to learn how to solve systems of equations that are not necessarily all linear. We call these non-linear systems of equations.

**Definition:** Non-linear system of equations
A system of equations where one or more equations involved is not a line.

We primarily use the substitution method to solve a non-linear system. However, sometimes elimination will work as well.

Also, just as before, the solution to a non-linear system is all the points of intersection of the graphs of the equations. Therefore, since we now have more that just lines, we can have a variety of numbers of solutions. The graph can intersect once, twice, several times or not at all. Therefore, we should always verify the solution(s) to a system by looking at the graph.

**Example 1:**

Solve the system.

\[
\begin{align*}
\text{a. } & x^2 + y^2 = 100 \\
& y - x = 2 \\
\text{b. } & x^2 + y^2 = 25 \\
& y^2 = x + 5 \\
\text{c. } & x^2 + 2y^2 = 12 \\
& xy = 4
\end{align*}
\]

**Solution:**

a. We will solve this system by substitution. So we start by solving the bottom equation for \( y \) and then substitute it into the top equation. We get

\[
\begin{align*}
& y - x = 2 \implies y = x + 2 \\
& x^2 + y^2 = 100 \implies x^2 + (x + 2)^2 = 100 \\
& x^2 + x^2 + 4x + 4 = 100 \\
& 2x^2 + 4x - 96 = 0 \\
& 2(x^2 + 2x - 48) = 0 \\
& 2(x + 8)(x - 6) = 0 \\
& x = -8, 6
\end{align*}
\]

So we have two different \( x \) values. This means we should have two points of intersection. Let's now find the \( y \) values and verify with a graph.

\[
\begin{align*}
& y = x + 2 \\
& y = -8 + 2 \\
& y = -6 \\
& y = 6 + 2 \\
& y = 8
\end{align*}
\]

So our solutions are \((-8, -6)\) and \((6, 8)\). We clearly have a circle and a line, thus we can easily graph them together and get
b. Again we will use substitution to solve. This time notice that the bottom equation is already solved for $y^2$ and we have a $y^2$ in the top equation. Thus, that is the substitution we will make. We get
\[
x^2 + y^2 = 25 \\
\Rightarrow x^2 + (x + 5) = 25
\]
Now we solve for $x$ and then solve for $y$.
\[
x^2 + (x + 5) = 25 \\
x^2 + x - 20 = 0 \\
(x + 5)(x - 4) = 0
\]
\[
x = -5, 4
\]
We substitute these back in to get $y$. We get
\[
y^2 = -5 + 5 \\
y^2 = 4 + 5 \\
y^2 = 0 \\
y = 0
\]
\[
y = \pm 3
\]
So we have three different solutions $(-5, 0), (4, 3)$ and $(4, -3)$. Lets verify with a graph. We have here a parabola and a circle. We get

\[
\text{c. Again, we will use substitution to solve. We need to decide which variable to solve for first. It seems that $x$ or $y$ on the bottom equation would be easiest. So we will solve for $x$. We get}
\]
\[
x^2 + y^2 = 25 \Rightarrow x = \frac{4}{y}
\]
Now we substitute that into the first equation and solve. We have
\[
x^2 + 2y^2 = 12 \Rightarrow \left(\frac{4}{y}\right)^2 + 2y^2 = 12
\]
\[
\frac{16}{y^2} + 2y^2 = 12
\]
We will have to clear the fractions and solve as we did in chapter 10, that is, using a substitution.
\[
y^2\left(\frac{16}{y^2} + 2y^2\right) = y^2(12)
\]
\[
16 + 2y^4 = 12y^2
\]
\[
2y^4 - 12y^2 + 16 = 0
\]
Substitute \(u = y^2\)
\[
2u^2 - 12u + 16 = 0
\]
\[
2(u^2 - 6u + 8) = 0
\]
\[
2(u - 4)(u - 2) = 0
\]
\[
u = 4, 2
\]
Re-substitute \(u = y^2\)
\[
y^2 = 4
\]
\[
y^2 = 2
\]
\[
y = \pm 2
\]
\[
y = \pm \sqrt{2}
\]
So since we have four different \(y\) values we will have to find \(x\) for each one. We substitute these in to \(x = \frac{4}{y}\) to get
\[
x = \frac{4}{2} \quad x = \frac{4}{-2} \quad x = \frac{4}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \quad x = \frac{4}{-\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}
\]
\[
= 2 
\]
\[
= -2 
\]
\[
= \frac{4\sqrt{2}}{2} = 2\sqrt{2}
\]
\[
= \frac{-4\sqrt{2}}{2} = -2\sqrt{2}
\]
So we have four solutions, \((2, 2), (-2, -2), (2\sqrt{2}, \sqrt{2})\) and \((-2\sqrt{2}, -\sqrt{2})\).
Lets verify with a graph. We have an ellipse and a basic function \((xy = 4 \Rightarrow y = \frac{4}{x})\).
So we have

Example 2:

Solve the system.

a. \(y = 3^x\)  
   \(y = 3^{2x} - 2\)

b. \(y = \log_2(x + 1)\)  
   \(y = 5 - \log_2(x - 3)\)
Solution:

a. This time solving is a little more complicated. There are a variety of directions we could go, however, we are going to start by noticing \( 3^{2x} = \left(3^x\right)^2 \). So our system really is

\[
\begin{align*}
y &= 3^x \\
y &= \left(3^x\right)^2 - 2
\end{align*}
\]

We can see clearly we will substitute the top equation into the bottom. That is, put \( y \) in for \( 3^x \). We get

\[
\begin{align*}
y &= (y)^2 - 2 \\
y &= y^2 - 2 \\
y^2 - y - 2 &= 0 \\
(y - 2)(y + 1) &= 0 \\
y &= 2, -1
\end{align*}
\]

Now we substitute these values back in to get

\[
\begin{align*}
2 &= 3^x \\
-1 &= 3^x
\end{align*}
\]

However, the second equation is impossible (recall, exponential functions are always positive). Thus, that solution must be omitted. So we have a solution of \((\log_2 3, 2)\).

b. Lastly, since these equations are both already solved for \( y \), we can simply set them equal to one another. Then we are left with a logarithmic equation to solve. We get

\[
\log_2(x + 1) = 5 - \log_2(x - 3)
\]

\[
\begin{align*}
\log_2(x + 1) + \log_2(x - 3) &= 5 \\
\log_2(x + 1)(x - 3) &= 5 \\
(x + 1)(x - 3) &= 2^5 \\
x^2 - 2x - 3 &= 32 \\
x^2 - 2x - 35 &= 0 \\
(x - 7)(x + 5) &= 0 \\
x &= 7, -5
\end{align*}
\]

However, -5 cannot be a solution since is doesn’t even check in the equation. Thus we only have \( x = 7 \). Now we substitute this back into either original equation to get the \( y \) value. We choose the first equation.

\[
\begin{align*}
y &= \log_2(7 + 1) \\
&= \log_2 8 \\
&= 3
\end{align*}
\]

So the solution is \((7, 3)\).

### 13.4 Exercises

Solve the systems.
1. \( x^2 + y^2 = 2 \)
   \( x + y = 2 \)

2. \( x^2 + y^2 = 25 \)
   \( y - x = 1 \)

3. \( 25x^2 + 9y^2 = 225 \)
   \( 5x + 3y = 15 \)

4. \( 9x^2 + 4y^2 = 36 \)
   \( 3x + 2y = 6 \)

5. \( y^2 = x + 3 \)
   \( 2y = x + 4 \)

6. \( y = x^2 \)
   \( 3x = y + 2 \)

7. \( x^2 - y^2 = 16 \)
   \( x - 2y = 1 \)

8. \( x^2 + 4y^2 = 25 \)
   \( x + 2y = 7 \)

9. \( x^2 + y^2 = 18 \)
   \( 2x + y = 3 \)

10. \( x^2 - y = 3 \)
    \( 2x - y = 3 \)

11. \( x^2 + y^2 = 20 \)
    \( y = x^2 \)

12. \( x^2 - y^2 = 3 \)
    \( y = x^2 - 3 \)

13. \( x^2 - x - y = 2 \)
    \( 4x - 3y = 0 \)

14. \( x^2 - 2x + 2y^2 = 8 \)
    \( 2x + y = 6 \)

15. \( x^2 + y^2 = 13 \)
    \( y = x^2 - 1 \)

16. \( x^2 - y = 5 \)
    \( x^2 + y^2 = 25 \)

17. \( x^2 + y^2 = 25 \)
    \( 2x^2 - 3y^2 = 5 \)

18. \( x^2 + y^2 = 4 \)
    \( 9x^2 + 16y^2 = 144 \)

19. \( x^2 + y^2 = 13 \)
    \( x^2 - y^2 = -16 \)

20. \( x^2 + y^2 = 16 \)
    \( y^2 - 2y^2 = 10 \)

21. \( x^2 + y^2 = 20 \)
    \( x^2 - y^2 = -12 \)

22. \( x^2 + y^2 = 14 \)
    \( x^2 - y^2 = 4 \)

23. \( xy = -\frac{9}{5} \)
    \( 3x + 2y = 6 \)

24. \( x + y = -6 \)
    \( xy = -7 \)

25. \( y = x^2 - 4 \)
    \( x^2 + y^2 = 16 \)

26. \( x^2 + y^2 = 25 \)
    \( y^2 = x + 5 \)

27. \( xy = \frac{1}{5} \)
    \( y + x = 5xy \)

28. \( xy = \frac{1}{12} \)
    \( x + y = 7xy \)

29. \( x^2 + xy + 2y^2 = 7 \)
    \( x - 2y = 5 \)

30. \( x^2 - xy + 3y^2 = 27 \)
    \( x - y = 2 \)

31. \( x^2 + y^2 = 5 \)
    \( xy = 2 \)

32. \( x^2 + y^2 = 20 \)
    \( xy = 8 \)

33. \( 3xy + x^2 = 34 \)
    \( 2xy - 3x^2 = 8 \)

34. \( 2xy + 3y^2 = 7 \)
    \( 3xy - 2y^2 = 4 \)

35. \( \frac{1}{x} + \frac{1}{y} = 5 \)
    \( \frac{1}{x} - \frac{1}{y} = -3 \)

36. \( \frac{1}{x} - \frac{1}{y} = 4 \)
    \( \frac{1}{x} + \frac{1}{y} = -2 \)
37. \( \frac{2}{x^2} + \frac{5}{y^2} = 3 \)  
38. \( \frac{1}{x^2} - \frac{3}{y^2} = 14 \)  
39. \( x^4 = y - 1 \)  
40. \( x^3 - y = 0 \)  
41. \( y = -\sqrt{x} \)  
42. \( y = \sqrt{x} \)  
43. \( \frac{3}{x^2} - \frac{2}{y^2} = 1 \)  
44. \( (x - 3)^2 + y^2 = 4 \)  
45. \( y = 5^x \)  
46. \( y = 2e^{2x} \)  
47. \( y = 2 - \log_2(x + 1) \)  
48. \( y = e^{4x} \)  
49. \( y = e^x - 1 \)  
50. \( y = \log_9(x + 1) \)  
60. \( y = \log_9(x + \frac{1}{2}) \)