3.6 Additional Topics With Rational Numbers

In this section we will consider some additional topics, and revisit some old topics, involving rational numbers. Our first topic involves ordering and comparing the size of rational numbers.

Which fraction is larger: \(\frac{4}{7}\) or \(\frac{7}{13}\)? Recall in Section 3.1 we answered this question by converting to decimal form. An alternate approach is to build each fraction to a common denominator, which is \(7\cdot13 = 91\). Converting each fraction:

\[
\begin{align*}
\frac{4}{7} \cdot \frac{13}{13} &= \frac{52}{91} \\
\frac{7}{13} \cdot \frac{7}{7} &= \frac{49}{91}
\end{align*}
\]

Since \(\frac{52}{91} > \frac{49}{91}\), then \(\frac{4}{7} > \frac{7}{13}\). This approach is usually faster than converting to decimal, especially as the denominators of the fractions become larger.

**Example 1**  Replace the blank with the correct symbol: <, =, or >

a. \(\frac{5}{11} \_ \frac{7}{16}\)

b. \(\frac{15}{22} \_ \frac{21}{31}\)

c. \(\frac{5}{15} \_ \frac{5}{21}\)

d. \(\frac{7}{12} \_ \frac{9}{16}\)

**Solution**  

a. Building each fraction to a common denominator:

\[
\begin{align*}
\frac{5}{11} \cdot \frac{16}{16} &= \frac{80}{176} \\
\frac{7}{11} \cdot \frac{11}{11} &= \frac{77}{176}
\end{align*}
\]

Since \(\frac{80}{176} > \frac{77}{176}\), then \(\frac{5}{11} > \frac{7}{16}\).
b. Building each fraction to a common denominator:

\[
\frac{15}{22} \cdot \frac{31}{31} = \frac{465}{682} \\
\frac{21}{31} \cdot \frac{22}{22} = \frac{462}{682}
\]

Since \( \frac{465}{682} < \frac{462}{682} \), then \( \frac{15}{22} < \frac{21}{31} \).

c. Building each fraction to a common denominator:

\[
\frac{13}{15} \cdot \frac{7}{7} = \frac{91}{105} \\
\frac{17}{21} \cdot \frac{5}{5} = \frac{85}{105}
\]

Since \( \frac{91}{105} > \frac{85}{105} \), then \( \frac{13}{15} > \frac{17}{21} \) and so \( \frac{5}{13} > \frac{5}{17} \).

d. Building each fraction to a common denominator:

\[
\frac{7}{12} \cdot \frac{4}{4} = \frac{28}{48} \\
\frac{9}{16} \cdot \frac{3}{3} = \frac{27}{48}
\]

Since \( \frac{28}{48} < \frac{27}{48} \), then \( \frac{7}{12} < \frac{9}{16} \).

Not only can we determine the order of rational numbers, we can represent this order by means of graphing inequalities. Recall that the number line is ordered by the < symbol; that is, as numbers which are smaller appear to the left of numbers which are larger. Suppose we were presented with the inequality \( x < 4 \). We can use a number line to “shade” those numbers which are less than 4 (remember, that means “to the left of” 4). Note the use of the open circle (○) at the number 4, indicating that 4 is not part of the shaded portion.

Now consider the inequality \( x \geq -2 \). We now must shade those numbers greater than or equal to –2, which means numbers “to the right of” –2. Note the use of the closed circle (●) at the number –2, indicating that –2 is part of the shaded portion.
Example 2  Graph the given inequality.

a. \( x > -5 \)

b. \( x < 2 \)

c. \( x \geq -2 \frac{3}{4} \)

d. \( x \leq -3.6 \)

Solution  a. Starting at \(-5\), shade all values to the right of \(-5\). Since the inequality is \(>\), we use the open circle at \(-5\), indicating it is not included in the shading:

b. Starting at \(2\), shade all values to the left of \(2\). Since the inequality is \(<\), we use the open circle at \(2\), indicating it is not included in the shading:

c. Starting at \(-2 \frac{3}{4}\), shade all values to the right of \(-2 \frac{3}{4}\). Since the inequality is \(\geq\), use the closed circle at \(-2 \frac{3}{4}\), indicating it is included in the shading:

d. Starting at \(-3.6\), shade all values to the left of \(-3.6\). Since the inequality is \(\leq\), use the closed circle at \(-3.6\), indicating it is included in the shading:

Recall in the last chapter we considered integer solutions to equations. We now consider rational number solutions to equations. Recall that a number is a solution to an equation if, after replacing the variable in the equation with the number, a true statement results. The following example will illustrate this process.
Example 3  Determine whether or not the given rational number is a solution to the equation.

a.  \( 2x + 3 = -2 ; \quad x = -\frac{5}{2} \)

b.  \( 3x + 1 = 6 ; \quad x = -\frac{5}{3} \)

c.  \( \frac{1}{2}x + \frac{1}{3} = -1 ; \quad y = -\frac{8}{3} \)

d.  \( \frac{a}{4} + 1 = \frac{a}{9} + \frac{1}{4} ; \quad a = -\frac{9}{2} \)

Solution  a.  Substitute \( x = -\frac{5}{2} \) into the equation evaluate the left side:

\[
2 \left( -\frac{5}{2} \right) + 3 = -2 \\
-5 + 3 = -2 \\
-2 = -2
\]

Since this last statement \((-2 = -2)\) is true, \( x = -\frac{5}{2} \) is a solution to the Equation \( 2x + 3 = -2 \).

b.  Substitute \( x = -\frac{5}{3} \) into the equation and evaluate the left side:

\[
3 \left( -\frac{5}{3} \right) + 1 = 6 \\
-5 + 1 = 6 \\
-4 = 6
\]

Since this last statement \((-4 = 6)\) is false, \( x = -\frac{5}{3} \) is not a solution to the Equation \( 3x + 1 = 6 \).
c. Substitute \( y = -\frac{8}{3} \) into the equation and evaluate the left side:

\[
\frac{1}{2} \left( -\frac{8}{3} \right) + \frac{1}{3} = -1
\]

\[
-\frac{4}{3} + \frac{1}{3} = -1
\]

\[
-\frac{3}{3} = -1
\]

\[-1 = -1\]

Since this last statement \((-1 = -1)\) is true, \( y = -\frac{8}{3} \) is a solution to the equation \( \frac{1}{2} y + \frac{1}{3} = -1 \).

d. Substitute \( a = -\frac{9}{2} \) into the equation and evaluate both sides:

\[
-\frac{9}{2} \cdot \frac{2}{4} + 1 = -\frac{9}{2} \cdot \frac{2}{2} \cdot \frac{2}{4} + 1
\]

\[
= -\frac{9}{8} + \frac{8}{8}
\]

\[
= -\frac{1}{8}
\]

\[
-\frac{9}{2} \cdot \frac{2}{2} + \frac{1}{4} = -\frac{9}{2} \cdot \frac{2}{2} \cdot \frac{2}{4} + \frac{1}{4}
\]

\[
= \frac{1}{8}
\]

\[
= -\frac{1}{4}
\]

Since \(-\frac{1}{8} \neq -\frac{1}{4}\), the two sides do not evaluate to the same number, so \( a = -\frac{9}{2} \) is not a solution to the equation \( \frac{a}{4} + 1 = \frac{a}{9} + \frac{1}{4} \).

Another topic studied in the last chapter was that of **arithmetic and geometric sequences**. Recall that an arithmetic sequence is one in which each term is the result of adding a common difference onto the previous term, while a geometric sequence is one in which each term is the result of multiplying a common ratio by the previous term. Now we consider sequences in which rational numbers are involved.
Example 4  

Given the sequence, determine whether it is arithmetic or geometric, find the common difference or common ratio, and find the next term.

a. \[-\frac{3}{2}, -\frac{3}{4}, -\frac{3}{8}, \ldots\]

b. \[2, \frac{2}{3}, \frac{2}{9}, \ldots\]

c. \[-2, -1.6, -1.2, \ldots\]

d. \[3, -0.6, 0.12, \ldots\]

Solution

a. First we need to determine if the sequence is arithmetic or geometric.

Note that:

\[
-\frac{3}{8} + \left(-\frac{3}{4}\right) = -\frac{3}{8} \cdot \left(-\frac{4}{3}\right) = \frac{2 \cdot 2 \cdot 3}{2 \cdot 2 \cdot 2 \cdot 3} = \frac{1}{2}
\]

\[
-\frac{3}{4} + \left(-\frac{3}{2}\right) = -\frac{3}{4} \cdot \left(-\frac{2}{3}\right) = \frac{2 \cdot 3}{2 \cdot 2 \cdot 3} = \frac{1}{2}
\]

So the sequence is geometric and the common ratio is \(\frac{1}{2}\). The next term is given by:

\[-\frac{3}{8} \cdot \frac{1}{2} = -\frac{3}{16}\]

b. First we need to determine if the sequence is arithmetic or geometric.

Note that:

\[
\frac{2}{3} - \frac{4}{3} = \frac{2}{3} + \left(-\frac{4}{3}\right) = -\frac{2}{3}
\]

\[
\frac{4}{3} - 2 = \frac{4}{3} + \left(-\frac{6}{3}\right) = -\frac{2}{3}
\]

So the sequence is arithmetic and the common difference is \(-\frac{2}{3}\). The next term is given by:

\[\frac{2}{3} + \left(-\frac{2}{3}\right) = 0\]
c. First we need to determine if the sequence is arithmetic or geometric. Note that:
\[-1.2 - (-1.6) = -1.2 + 1.6 = 0.4\]
\[-1.6 - (-2) = -1.6 + 2 = 0.4\]
So the sequence is arithmetic and the common difference is 0.4. The next term is given by:
\[-1.2 + 0.4 = -0.8\]

d. First we need to determine if the sequence is arithmetic or geometric. Note that:
\[0.12 \div (-0.6) = -0.2\]
\[-0.6 \div 3 = -0.2\]
So the sequence is geometric and the common ratio is \(-0.2\). The next term is given by:
\[0.12 \cdot (-0.2) = -0.024\]

We conclude this section with an application of rational numbers which involves order of operations.

**Example 5**  
Steve owns 120 shares of a stock which rises \(\$3 \frac{1}{2}\) per share one day, and 250 shares of a stock which loses \(\$1 \frac{5}{8}\) per share on the same day. Did he have a net gain or loss, and how much was it?

**Solution**  
Treating the stock rise as a positive number and the stock loss as a negative number, his net gain or loss is given by:
\[
120 \left( 3 \frac{1}{2} \right) + 250 \left( -1 \frac{5}{8} \right) = 120 \cdot \frac{7}{2} + 250 \cdot \left( -\frac{13}{8} \right)
\]
\[
= 420 + (-406.25)
= 13.75
\]
Steve had a slight gain of \$13.75 on the day. Note that we converted to decimals in the second step, since they are commonly used in our dollar money system.

**Terminology**

- graphing inequalities
- arithmetic sequences
- geometric sequences
Exercise Set 3.6

Replace the blank with the correct symbol: <, =, or >

1. \[
\frac{7}{12} \quad \frac{14}{25} \\
\frac{10}{7} \quad \frac{27}{18}
\]

2. \[
\frac{19}{36} \quad \frac{27}{50} \\
\frac{53}{17} \quad \frac{64}{20}
\]

3. \[
\frac{16}{23} \quad \frac{29}{47} \\
\frac{25}{10} \quad \frac{56}{21}
\]

4. \[
\frac{13}{30} \quad \frac{19}{45} \\
\frac{24}{11} \quad \frac{35}{14}
\]

5. \[
\frac{4}{9} \quad \frac{3}{4} \\
-\frac{7}{9} \quad -\frac{13}{24}
\]

6. \[
\frac{15}{26} \quad \frac{36}{65}
\]

7. \[
-\frac{9}{16} \quad -\frac{13}{24}
\]

8. \[
\frac{7}{11} \quad \frac{71}{15}
\]

9. \[
\frac{5}{8} \quad \frac{18}{5}
\]

Graph the given inequality.

15. \(x > -3\)
16. \(x > 5\)
17. \(x \geq 4\)
18. \(x \geq -7\)
19. \(x < 5\)
20. \(x < 7\)
21. \(x \leq 1\)
22. \(x \leq -5\)
23. \(x > -2\frac{2}{3}\)
24. \(x > 4\frac{3}{4}\)
25. \(x \geq -\frac{19}{3}\)
26. \(x \geq -\frac{12}{5}\)
27. \(x < -\frac{25}{6}\)
28. \(x < -\frac{23}{8}\)
29. \(x \leq 5\frac{1}{8}\)
30. \(x \leq -6\frac{1}{3}\)
31. \(x \leq -4.7\)
32. \(x \leq 5.3\)
33. \(x > 10.4\)
34. \(x > -3.7\)
35. \(x \geq -0.6\)
36. \(x \geq -0.99\)
Determine whether or not the given rational number is a solution to the equation.

37. \(3x + 4 = -1; \ x = -\frac{5}{3}\)
39. \(2y - 5 = -2; \ y = -\frac{3}{2}\)
41. \(2x - 1 = 4x - 6; \ x = \frac{5}{2}\)
43. \(-3x - 5 = 2x - 4; \ x = \frac{1}{5}\)
45. \(\frac{2}{3}z - 1 = 4; \ z = -\frac{15}{2}\)
47. \(-\frac{1}{2}a - 1 = -\frac{1}{4}; \ a = -\frac{3}{2}\)
49. \(-\frac{3}{4}x - 1 = \frac{1}{4}x - 3; \ x = -2\)
51. \(\frac{w}{3} + 1 = -\frac{w}{6} + \frac{3}{4}; \ w = -\frac{1}{2}\)

38. \(3x + 4 = -1; \ x = \frac{5}{3}\)
40. \(2y - 5 = -2; \ y = \frac{3}{2}\)
42. \(2x - 1 = 4x - 6; \ x = -\frac{5}{2}\)
44. \(-3x - 5 = 2x - 4; \ x = -\frac{1}{5}\)
46. \(\frac{2}{3}z - 1 = 4; \ z = \frac{15}{2}\)
48. \(-\frac{1}{2}a - 1 = -\frac{1}{4}; \ a = \frac{3}{2}\)
50. \(-\frac{3}{4}x - 1 = \frac{1}{4}x - 3; \ x = 2\)
52. \(\frac{w}{3} + 1 = -\frac{w}{6} + \frac{3}{4}; \ w = \frac{1}{2}\)

Given the sequence, determine whether it is arithmetic or geometric, find the common difference or common ratio, and find the next term.

53. \(-\frac{2}{3}, -\frac{1}{3}, -\frac{1}{6},...\)
55. \(-4, -2, -1, ...\)
57. \(-3, -\frac{7}{3}, -\frac{5}{3}, ...\)
59. \(2, \frac{1}{2}, -1, ...\)
61. \(-12, 6, -3, ...\)
63. \(-\frac{5}{12}, \frac{5}{18}, -\frac{5}{27}, ...\)
65. \(-1.8, -1.4, -1, ...\)
67. \(1.5, 0.6, -0.3, ...\)
69. \(8, -1.6, 0.32, ...\)

54. \(-\frac{3}{4}, -\frac{1}{4}, -\frac{1}{12}, ...\)
56. \(-9, -3, -1, ...\)
58. \(-1, -\frac{1}{4}, -\frac{1}{2}, ...\)
60. \(1, \frac{1}{3}, -\frac{1}{3}, ...\)
62. \(12, -4, \frac{4}{3}, ...\)
64. \(\frac{9}{14}, -\frac{3}{7}, -\frac{2}{7}, ...\)
66. \(-3.4, -2.3, -1.2, ...\)
68. \(2.3, 0.9, -0.5, ...\)
70. \(-5, 1.5, -0.45, ...\)
Answer each of the following application questions. Be sure to read the question, interpret the problem mathematically, solve the problem, then answer the question. You should answer the question in the form of a sentence.

71. Bernice owns 460 shares of a stock which gains \(\frac{3}{8}\) per share one day, and 320 shares of a stock which loses \(\frac{1}{2}\) per share that day. Did she have a net gain or loss that day, and how much was it?

72. Brian owns 360 shares of a stock which gains \(\frac{7}{8}\) per share one day, and 170 shares of a stock which loses \(\frac{3}{4}\) per share that day. Did he have a net gain or loss that day, and how much was it?

73. Todd signs a two year lease for his new BMW. The lease requires a $2,450 down payment and payments of $473.58 per month. What is the total amount he paid for the lease?

74. Brad signs a four year loan for his yacht. The loan requires a $6,500 down payment and payments of $683.97 per month. What is the total amount he paid for the yacht?

75. Carol signs a 15 year loan for her house. The loan requires a $18,760 down payment and payments of $1,265.49 per month. What is the total amount she paid for the house?

76. Tracy signs a 30 year loan for her house. The loan requires a $6,890 down payment and payments of $732.26 per month. What is the total amount she paid for the house?

77. Jerry’s walnut trees produce 80 pounds of walnuts per tree. He plants 50 trees per acre, and has 86 acres of walnuts. If Jerry is paid $0.67 per pound for the walnuts, what is the total revenue from his walnut orchard?

78. During a low production year, Jerry (from Exercise 77) has a yield of only 62 pounds of walnuts per tree. However, the price paid for the walnuts raises to $0.84 per pound. Will his total revenue increase or decrease? By how much?

79. You commute to (and from) work 218 times during the year. The distance from your home to work is 24 miles. If the cost of operating your car is 26 cents/mile, what is your cost of commuting during the year?

80. Mary commutes to (and from) work 209 times during the year. The distance from her home to work is 8 miles. If the cost of operating her car is 35 cents/mile, what is her cost of commuting during the year?

81. Frank’s Cleaners contracts to clean 325 shirts and 130 pairs of pants for one week. Frank is paid $2.25 for each shirt and $4.25 for each pair of pants. How much is the weekly contract?
82. A separate contract with Frank’s Cleaners (from Exercise 81) is to clean 286 shirts and 155 pairs of pants for one week. If the amount per shirt and pair of pants remains unchanged, will this contract be more or less? By how much?