2.6 Exponents and Order of Operations

We begin this section with exponents applied to negative numbers. The idea of applying an exponent to a negative number is identical to that of a positive number (repeated multiplication), thus

\[(−3)^2 = (−3)(−3) = 9\]

and

\[(−4)^3 = (−4)(−4)(−4) = −64\]

What happens if the parentheses are removed from the number? To compute \(−4^2\), the best way to understand its meaning is to note that

\[−4^2 = 0 − 4^2\]

Since order of operations must apply to this expression, the multiplication needs to be computed before the subtraction. Thus

\[−4^2 = 0 − 4^2 = 0 − 16 = −16\]

Commonly we read \(−4^2\) as the *opposite* of \(4^2\), rather than *negative* \(4^2\), which instead would be written as \((−4)^2\). As long as you keep this distinction in mind, exponents on negative numbers are computed exactly as on positive numbers.
Example 1  Compute each exponent.

a. \((-9)^2\)

b. \((-3)^4\)

c. \(-2^4\)

d. \((-5)^3\)

Solution  

a. Computing the exponent:
\((-9)^2 = (-9)(-9) = 81\)

b. Computing the exponent:
\((-3)^4 = (-3)(-3)(-3)(-3) = 81\)

c. Remembering this refers to the opposite:
\(-2^4 = -(2 \cdot 2 \cdot 2 \cdot 2) = -16\)

d. Computing the exponent:
\((-5)^3 = (-5)(-5)(-5) = -125\)

We are now ready to tackle order of operations. Recall the order of operations as:

Order of Operations

1. First compute all parentheses.
2. Compute all exponents next.
3. Compute all multiplications and divisions (working left to right).
4. Compute all additions and subtractions (working left to right).

These are identical to the last chapter, except integers will now be used instead of whole numbers. The following examples should help to clarify the use of order of operations for computation with integers.
Example 2  Compute the following expressions.

a.  $-5 - 4(3 + 2)$

b.  $-12 - 8 + 2$

c.  $(3 - 7)(11 - 4^2)$

d.  $(-4)^2 - (-6)^2$

Solution  

a.  First compute within the parentheses first, then compute the multiplication, then finally the subtraction:

$$-5 - 4(3 + 2) = -5 - 4 \cdot 5$$
$$= -5 - 20$$
$$= -5 + (-20)$$
$$= -25$$

b.  First compute the division, then the subtraction:

$$-12 - 8 + 2 = -12 - 4$$
$$= -12 + (-4)$$
$$= -16$$

c.  First compute within the parentheses (remember to do the exponent first in the second parentheses), then compute the multiplication:

$$(3 - 7)(11 - 4^2) = (3 - 7)(11 - 16)$$
$$= (3 + (-7))(11 + (-16))$$
$$= (-4)(-5)$$
$$= 20$$

d.  First compute the exponents, then compute the subtraction:

$$(-4)^2 - (-6)^2 = (-4)(-4) - (-6)(-6)$$
$$= 16 - 36$$
$$= 16 + (-36)$$
$$= -20$$

Note how we still rewrite subtraction as an addition problem.
Example 3  Compute the following expressions.

a. \( (4 - 2^3) - 3 \cdot 2 \)

b. \( (4 - 9)^2 - (5 - 8)^3 \)

c. \( (4 - (-3)^2)^2 \)

d. \(-\left(36 + 2^2\right)^2 - 48 + (-4)^2\)

Solution  

a. Compute within the parentheses first (remember to compute the exponent first, then compute the multiplication, then finally the subtraction (rewriting as an addition):

\[
4 - 2^3 - 3 \cdot 2 = (4 - 8) - 3 \cdot 2 \\
= (4 + (-8)) - 3 \cdot 2 \\
= -4 - 3 \cdot 2 \\
= -4 - 6 \\
= -4 + (-6) \\
= -10
\]

b. Compute within each parentheses first, then compute the exponents, and finally the subtraction (rewriting as an addition):

\[
(4 - 9)^2 - (5 - 8)^3 = (4 + (-9))^2 - (5 + (-8))^3 \\
= (-5)^2 - (-3)^3 \\
= 25 - (-27) \\
= 25 + 27 \\
= 52
\]

c. Compute within the parenthesis first (remember to compute the exponent first, then the subtraction), then finally compute the outside exponent:

\[
(4 - (-3)^2)^2 = (4 - 9)^2 \\
= (4 + (-9))^2 \\
= (-5)^2 \\
= 25
\]
d. Compute within the parenthesis first (exponent first, then the division), then compute the exponents outside of the exponents, then the division, and finally the subtraction (rewriting as an addition):

$$\left( 36 + 2^2 \right)^2 - 48 + (-4)^2 = \left( 36 + 4 \right)^2 - 48 + (-4)^2$$

$$= (9)^2 - 48 + (-4)^2$$

$$= -81 - 48 + 16$$

$$= -81 - 3$$

$$= -81 + (-3)$$

$$= -84$$

Armed with order of operations, we are now able to check if integers are solutions to more complicated equations. For example, to determine if $x = -3$ is a solution to the equation $2x - 1 = -7$, we substitute into the equation and determine if a true statement exists:

$$2(-3) - 1 = -7$$

$$-6 - 1 = -7$$

$$-6 + (-1) = -7$$

$$-7 = -7$$

Since this statement ($-7 = -7$) is true, $x = -3$ is a solution to the equation $2x - 1 = -7$. Note that there are more steps to the check, especially if the variable appears on both sides of the equation.
Example 4  Determine whether or not the given integer value is a solution to the equation.

a. \(3x + 5 = -2; x = -1\)

b. \(-2y - 3 = 1; y = -2\)

c. \(-4a - 3 = -11; a = -2\)

d. \(2x + 1 = 4x - 5; x = 3\)

Solution  a. Substitute \(x = -1\) into the equation and determine whether the statement is true:

\[
3(-1) + 5 = -2 \\
-3 + 5 = 2 \\
2 = 2
\]

Since this last statement \((2 = 2)\) is true, \(x = -1\) is a solution to the equation \(3x + 5 = -2\).

b. Substitute \(y = -2\) into the equation and determine whether the statement is true:

\[
-2(-2) - 3 = 1 \\
4 - 3 = 1 \\
1 = 1
\]

Since this last statement \((1 = 1)\) is true, \(y = -2\) is a solution to the equation \(-2y - 3 = 1\).

c. Substitute \(a = -2\) into the equation and determine whether the statement is true:

\[
-4(-2) - 3 = -11 \\
8 - 3 = -11 \\
5 = -11
\]

Since this last statement \((5 = -11)\) is false, \(a = -2\) is not a solution to the equation \(-4a - 3 = -11\).

d. Substitute \(x = 3\) into both sides of the equation, computing each side independently, and determine whether the statement is true:

\[
2(3) + 1 = 4(3) - 5 \\
6 + 1 = 12 - 5 \\
7 = 7
\]

Since this last statement \((7 = 7)\) is true, \(x = 3\) is a solution to the equation \(2x + 1 = 4x - 5\).
Terminology

order of operations

Exercise Set 2.6

Compute each exponent.

1. \((-6)^2\)  
2. \(-6^2\)  
3. \(-10^2\)  
4. \((-10)^2\)  
5. \((-3)^3\)  
6. \(-3^3\)  
7. \(-2^5\)  
8. \((-2)^5\)  
9. \(-(-2)^4\)  
10. \(-(-2)^6\)

Compute the following expressions. Remember to use order of operations in computing the values.

11. \(3 \cdot 4 - 5(8 - 2)\)  
12. \(5 \cdot 6 - 7(10 - 3)\)  
13. \(3 \cdot 4^2 - 4 \cdot 3^2\)  
14. \(6 \cdot 3^2 - 8 \cdot 4^2\)  
15. \(-12 - 4(4 - 5 \cdot 3)\)  
16. \(-15 - 5(3 - 4 \cdot 3)\)  
17. \((-8)(-3) - 4(-6)\)  
18. \((-9)(-3) - 5(-7)\)  
19. \((-5)^2 - (-6)^2\)  
20. \((-8)^2 - (-9)^2\)  
21. \(-12(-2)^2 - 5(-3)^3\)  
22. \(-8(-2)^4 - 6(-4)^2\)  
23. \(8 - 12(5 - 13)\)  
24. \(6 - 15(6 - 11)\)  
25. \(13 - 8(9 - 4)\)  
26. \(15 - 7(12 - 6)\)  
27. \(22 - 13(9 - 2^2)\)  
28. \(15 - 7(15 - 3^2)\)  
29. \((-6)^2 - 8(3 - 9)\)  
30. \((-5)^2 - 11(2 - 7)\)  
31. \(|-5|^2 - 4(3 - |-9|)\)  
32. \(|-6|^2 - 8(7 - |-12|)\)  
33. \(-6(-4 - 7) + 5(3 - 9)\)  
34. \(-5(-5 - 4) + 4(2 - 11)\)  
35. \(-(9 - 4^2)^2\)  
36. \(-(14 - 5^2)^2\)  
37. \((6 - 8)^2 - (4 - 7)^3\)  
38. \((4 - 9)^2 - (-3 - 2)^3\)  
39. \((-6 - 2)^2 - (-4 + 2)^3\)  
40. \((-3 - 2)^3 - (-5 + 2)^2\)
41. \( (6 - (-3)^2)^2 \)
43. \( (12 - 3^3) - 12 \cdot 5 \)
45. \( -(18 + 3^2)^2 - 54 + (-3)^2 \)
47. \( \frac{(-4)^2 - 3^2}{5 - 3 \cdot 4} \)
49. \( \frac{-3(13 - 15) - 8(5 - 3)}{4(-2) - 2 - (-5)} \)

42. \( (5 - (-4)^2)^2 \)
44. \( (42 - 4^3) - 6 \cdot 8 \)
46. \( -(100 + 5^2)^2 - 72 + (-6)^2 \)
48. \( \frac{5^2 - (-3)^2}{4 - 2 \cdot 4} \)
50. \( \frac{-4(4 - 6) - 4(2 - 3)}{-4 - 2(-5)} \)

Determine whether or not the given integer value is a solution to the equation.

51. \( 5x - 3 = 7 \); \( x = 2 \)
53. \( 4a + 1 = -15 \); \( a = 4 \)
55. \( -3x + 2 = -16 \); \( x = 6 \)
57. \( -8x + 1 = 17 \); \( x = 2 \)
59. \( 5y + 3 = 4y - 1 \); \( y = -4 \)
61. \( 6x - 5 = 4x - 11 \); \( x = 3 \)

63. \( \frac{t}{2} - 4 = \frac{t}{3} - 3 \); \( t = 6 \)
65. \( \frac{s}{3} - 6 = \frac{s}{4} + 1 \); \( s = 12 \)
67. \( x^2 = 4x \); \( x = 4 \)
69. \( w^2 - 5w = w - 8 \); \( w = -2 \)

52. \( 5x - 3 = 7 \); \( x = -2 \)
54. \( 4a + 1 = -15 \); \( a = -4 \)
56. \( -3x + 2 = -16 \); \( x = -6 \)
58. \( -8x + 1 = 17 \); \( x = -2 \)
60. \( 5y + 3 = 4y - 1 \); \( y = 4 \)
62. \( 6x - 5 = 4x - 11 \); \( x = -3 \)

64. \( \frac{t}{2} - 4 = \frac{t}{3} - 3 \); \( t = -6 \)
66. \( \frac{s}{3} - 6 = \frac{s}{4} + 1 \); \( s = -12 \)
68. \( x^2 = 4x \); \( x = -4 \)
70. \( w^2 - 5w = w - 8 \); \( w = 2 \)

71. What number must be subtracted from \(-8\) to yield \(-23\) ?
72. What number must be subtracted from \(4\) to yield \(12\) ?
73. What number must be divided by \(-4\) to yield \(-9\) ?
74. What number must be divided by \(6\) to yield \(-25\) ?
75. If the sum of \(-8\) and \(-4\) is divided by the sum of \(8\) and \(-10\), what is the result?
76. If the sum of \(-12\) and \(-13\) is divided by the sum of \(2\) and \(-7\), what is the result?
77. If the sum of \(-9\) and \(-12\) is divided by the difference of \(7\) and \(10\), what is the result?
78. If the difference of \(-8\) and \(20\) is divided by the difference of \(-2\) and \(5\), what is the result?
79. If the sum of \(-8\) and \(5\) is multiplied by the difference of \(5\) and \(14\), what is the result?
80. If the sum of \(-4\) and \(-9\) is multiplied by the difference of \(-6\) and \(13\), what is the result?